2019 Science Atlantic Math Competition

Time: 3 hours.

Team members may collaborate with each other but not with others. Calculators and notes are forbidden.

Please write the answer to each question on a separate sheet (or sheets) of paper, and do not refer to other answers, as your answers to the various questions will be graded separately. Put your **team number** and the **question number** on ALL pages. Do not put your names, team name, or university on the answer sheets. Show all work.

Put your university, your own names, and your team number on the outside of the envelope before handing in your answers.

Few marks will be given for fragmentary or incomplete answers, or correct answers without proof.

Each of the eight questions carries equal weight.

2019 Science Atlantic Math Competition

QUESTIONS

1. Show that the sum of five consecutive perfect squares is not itself a perfect square.

2. Let m and n be positive integers. Suppose there are m ants walking to the right and n ants walking to the left along a line. All ants walk at a constant speed and initially the right-walking ants are all to the left of the left-walking ants.

Whenever two ants collide, they immediately turn around and walk back in the opposite direction. (Of course, if an ant turns around, it may then collide with the ant that was following it and turn around again, and so on.) Find (as a function of m and n) the number of collisions that occur.

3. Squares are erected outward on all four sides of a parallelogram. Show that the centers of the squares themselves form a square.

4. Let $y = x + \frac{x}{x+\frac{x}{x+\frac{x}{x+\dots}}}$. For what (nonzero) integer values of x is y also an integer?

5. Let C be a unit cube. Given any face F of C, and any edge e of F, let $V_{F,e}$ be the reflection of the center of F in the edge e. Let V be the set of all points that can be obtained in this manner. Sketch the convex polyhedron which has V as its set of vertices, and describe it (with proof) in familiar terms.

6. Determine all triples of rational numbers (a, b, c) such that

$$a^3 + 2b^3 + 4c^3 = 6abc\ .$$

7. An $n \times n$ matrix $\mathbf{A}$ is said to have period p if p is the smallest positive integer such that $\mathbf{A}^p = \mathbf{I}_n$ (the $n \times n$ identity matrix). For what p is there a 2×2 matrix with integer elements that has period p?

8. Determine the following quantity or prove that it does not exist.

$$\lim_{n\to\infty} \left[\sum_{j=1}^{n} \frac{2^{\frac{j}{n}} - 2^{\frac{j-1}{n}}}{2^{\frac{j}{n}} + 1} \right]$$