2018 APICS Math Competition

Time: 3 hours.

Team members may collaborate with each other but not with others. Calculators and notes are forbidden.

Please write the answer to each question on a separate sheet (or sheets) of paper, and do not refer to other answers, as your answers to the various questions will be graded separately. Put your **team number** and the **question number** on ALL pages. Do not put your names, team name, or university on the answer sheets. Show all work.

Put your university, your own names, and your team number on the outside of the envelope before handing in your answers.

Few marks will be given for fragmentary or incomplete answers.

Each of the eight questions carries equal weight.

2016 APICS Math Competition

QUESTIONS

1. Prove that if p and q are consecutive odd primes, then $p+q$ is the product of at least three primes (not necessarily distinct.)

2. A *parallelopiped* is the 3-dimensional analogue of a parallelogram. You may think of it as a prism whose faces are parallelograms, none of which need be rectangular. Prove that the sum of the squares of the lengths of all edges of a parallelopiped equals the sum of the squares of the lengths of of all the body (=long) diagonals.

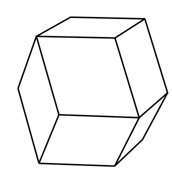
3. Determine whether not the following series converges. If it is convergent, give its sum explicitly.

$$\sum_{n=1}^{\infty}(\lfloor\frac{2n+7}{n+1}\rfloor)^{-n}$$

Note that $\lfloor x\rfloor$ is the greatest integer less than or equal to the real number x.

4. There are $n!$ permutations $(a_1, a_2, a_3, \ldots, a_n)$ of $(1, 2, 3, \ldots, n)$. How many of them satisfy $a_k \geq k-2$ for all $k = 1, 2, \ldots, n$?

5. A *rhombic dodecahedron* has twelve congruent rhombic faces; each vertex has either four small angles or three large angles meeting there. If the edge length is 1, find the volume in the form $\frac{p+\sqrt{q}}{r}$, where p, q, and r are natural numbers and r has no factor in common with p or q.



6. Let S be the set of natural numbers dividing 2018^{2018}. In how many ways can one select three numbers $\{x, y, z\}$ (not necessarily distinct, but order being irrelevant) from S so that $y=\sqrt{xz}$?

7. For any $n\times n$ matrix $[A]$, let $[A]^R$ be the matrix obtained by rotating $[A]$ ninety degrees in a clockwise direction. E.g.:

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix}^R = \begin{bmatrix} c & a \\ d & b \end{bmatrix}.$$

Find a formula for the determinant $\det[A]^R$ in terms of $\det[A]$ and n.

8. Find all solutions to the functional equation $f(f(x)) + f(x) = x$ that have continuous second derivative and such that $f(2018) > 0$.