2017 APICS Math Competition

Time: 3 hours.

Team members may collaborate with each other but not with others. Calculators and notes are forbidden.

Please write the answer to each question on a separate sheet (or sheets) of paper, and do not refer to other answers, as your answers to the various questions will be graded separately. Put your **team number** and the **question number** on ALL pages. Do not put your names, team name, or university on the answer sheets. Show all work.

Put your university, your own names, and your team number on the outside of the envelope before handing in your answers.

Few marks will be given for fragmentary or incomplete answers.

This question booklet has 8 questions over 2 pages. Each of the eight questions carries equal weight.

QUESTIONS

1. Find
$$\sum_{\substack{0\le j,k\\ j+k\le n}} x^j y^k$$
in closed form.

2. If x, y, n are positive integers such that
$$xy = 2^n + 1$$
prove that $x-1$ and $y-1$ are divisible by identical powers of 2.

3. Define the $n \times n$ *Pascal matrix* as follows: $a_{1j} = a_{i1} = 1$, while $a_{ij} = a_{i-1,j} + a_{i,j-i}$ for $i, j > 1$. So, for instance, the 3×3 Pascal matrix is
$$\begin{bmatrix} 1 & 1 & 1 \\ 1 & 2 & 3 \\ 1 & 3 & 6 \end{bmatrix}.$$
Show that every Pascal matrix is invertible.

4. For all positive integers n prove that

$$\frac{1}{n}+\frac{1}{n+1}+\frac{1}{n+2}+\cdots+\frac{1}{2n-1}=1-\frac{1}{2}+\frac{1}{3}-\frac{1}{4}+\cdots+\frac{1}{2n-1}.$$

5. Let $ABCD$ be a square, F the midpoint of DC, and E any point on AB such that $AE > EB$. Then locate H on BC such that $DE \| FH$.

   Prove that EH is tangent to the inscribed circle of the square.

6. Suppose $P(x)$ and $Q(x)$ are polynomials with real coefficients. Find necessary and sufficient conditions on N to guarantee that if the polynomial $P(Q(x))$ has degree N, there exists real x with $P(x) = Q(x)$.

7. Find (with proof) all integer solutions (x, y) to $x^2 - xy + 2017y = 0$.

8. On the University of Fredericton campus there are infinitely many gopher holes, evenly spaced in a line, although there is only one gopher. Every hour on the hour, the gopher moves from his current hole to another; he always moves in the same direction and always skips the same number of holes. Every ten minutes, Frederica the groundskeeper (who can walk arbitrarily fast) comes along and shines her flashlight in a gopher hole. She doesnt know where the gopher started, or how many holes the gopher skips. Find a strategy for her that will guarantee that she can find the gopher in a finite number of steps.