2016 APICS Math Competition

Time: 3 hours.

Team members may collaborate with each other but not with others. Calculators and notes are forbidden.

Please write the answer to each question on a separate sheet (or sheets) of paper, and do not refer to other answers, as your answers to the various questions will be graded separately. Put your **team number** and the **question number** on ALL pages. Do not put your names, team name, or university on the answer sheets. Show all work.

Put your university, your own names, and your team number on the outside of the envelope before handing in your answers.

Few marks will be given for fragmentary or incomplete answers.

Each of the eight questions carries equal weight.

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QUESTIONS

1 Let $ABCD$ be a tetrahedron (not necessarily regular.) Let M_{AB} be the midpoint of the edge $\overline{AB}$, and so on for the other five edges. Show that the lines $\overline{M_{AB}M_{CD}}$,$\overline{M_{AC}M_{BD}}$, and $\overline{M_{AD}M_{BC}}$ are concurrent (have a point common to all three.)

2 *The Hexagon Identity*: Pascal's Triangle is defined as follows: the boundary elements are all 1, each row is staggered a half-space from the one above it, and every internal element is the sum of the two diagonally above it. Every element not on a boundary has six neighbours in a hexagon:

$$
\begin{array}{ccccccccccccccc}
 & & & & & & & 1 & & & & & & & \\
 & & & & & & 1 & & 1 & & & & & & \\
 & & & & & 1 & & 2 & & 1 & & & & & \\
 & & & & 1 & & 3 & & 3 & & 1 & & & & \\
 & & & 1 & & 4 & & 6 & & 4 & & 1 & & & \\
 & & 1 & & [5] & & (10) & & 10 & & 5 & & 1 & & \\
 & 1 & & (6) & & \mathit{15} & & [20] & & 15 & & 6 & & 1 & \\
1 & & 7 & & [21] & & (35) & & 35 & & 21 & & 7 & & 1
\end{array}
$$

Prove that the product of three alternate neighbours, in this example $5 \times 20 \times 21$ (in parentheses), always equals the product of the other three, here, $10 \times 35 \times 6$ [in brackets.]

3 Find the volume of the portion of a unit cube that is at distance at most $\sqrt{2}$ from a specified corner.

4 Solve this system of simultaneous equations for real numbers x, y, z:

$$
\begin{array}{rcl}
x + \lfloor y \rfloor + \{z\} & = & 1.1 \\
\{x\} + y + \lfloor z \rfloor & = & 2.2 \\
\lfloor x \rfloor + \{y\} + z & = & 3.3
\end{array}
$$

Notation: $\lfloor x \rfloor$ is the *greatest integer less than or equal* x. On the other hand, $\{x\}$ here denotes the *the fractional part of* x. For example, $\{6.9018\} = .9018$, $\{42\} = 0$, $\{-\pi\} = 0.858407\ldots$.

5 Suppose $p(x)$ and $q(x)$ are two real polynomials such that

$$p(q(x)) = q(p(x))$$

for infinitely many values of x. If $p(x) = q(x)$ has no real solutions, prove that

$$p(p(x)) = q(q(x))$$

also has no real solutions.

6 A *rhumb line* or *loxodrome* is a curve on the earth's surface that follows a constant direction relative to true (not magnetic) north. Find the maximum possible length of a rhumb line directed northeastward (a bearing of 45° true), or show that it can be arbitrarily long. You may assume the earth to be a sphere of circumference 40,000 km.

7 Find (with proof) a closed form expression for

$$\sum_{i=0}^{\infty} \frac{(2i+1)^2}{2^i}$$

8 Big Sandy MacDonald and Little Sandy MacDonald take turns choosing positive integers to be the coefficients of a sixteenth degree polynomial

$$a_0 + a_1 z + \cdots a_{16} z^{16} \ .$$

The same coefficient may be used more than once. Little Sandy moves first, and wins the game if, at the end, the polynomial has a repeated root (real or complex), or two distinct real or complex roots ζ_1, ζ_2 with $|\zeta_1 - \zeta_2| \leq 1$. Find a winning strategy for Little Sandy and show that it works.