2015 APICS Math Competition

Time: 3 hours.

Team members may collaborate with each other but not with others. Calculators and notes are forbidden.

Please write the answer to each question on a separate sheet (or sheets) of paper, and do not refer to other answers, as your answers to the various questions will be graded separately. Put your **team number** and the **question number** on ALL pages. Do not put your names, team name, or university on the answer sheets. Show all work.

Put your university, your own names, and your team number on the outside of the envelope before handing in your answers.

Few marks will be given for fragmentary or incomplete answers.

Each of the eight questions carries equal weight.

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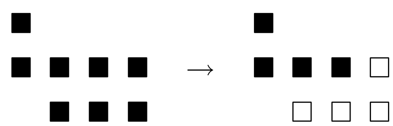
QUESTIONS

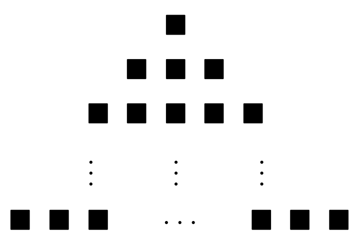
1. Determine all real numbers x such that

$$x - \sqrt{x} \cdot \log_{\frac{1}{\sqrt{2}}} x \geq \sqrt{x} - x \cdot \log_{\frac{1}{\sqrt{2}}} x$$

2. The following pattern of eight square tiles can be divided into two congruent sets of four tiles as shown. (Note that one set is the mirror image of the other - this is legal.)



Find a way to divide the following pattern of 100 tiles into two congruent sets of fifty tiles, or show it cannot be done.



(19 tiles in the 10th row)

3. Find all triples of continuous functions $f, g, h : \mathbf{R} \to \mathbf{R}$ such that, for all $x \in \mathbf{R}$,

$$f(g(x)) = g(h(x)) = h(f(x)) = x\ .$$

4. A line segment of length 1 moves continuously so that it is always tangent, at one end, to the ellipse $x^2 + 4y^2 = 1$. The other end traces out a curve K. Find the area inside K.

5. Prove that

$$1! \cdot 1 + 2! \cdot 2 + 3! \cdot 3 + \ldots + 2015! \cdot 2015 = 2016! - 1$$

6. How many of the first 2015 triangular numbers $\left\{ \frac{1 \cdot 2}{2}, \frac{2 \cdot 3}{2}, \cdots, \frac{2015 \cdot 2016}{2} \right\}$ are divisible by 2015?

7. Suppose movable points A, B lie on the positive x-axis and y-axis, respectively, in such a way that $\triangle ABO$ always has area 4. (The origin is point O.) Find an equation for a curve in the first quadrant which is tangent to each of the line segments AB.

8. Let $k_1 < k_2 < \cdots$ be positive integers, no two consecutive (*i.e.*, $k_n \geq k_{n-1} + 2$.) Let $s_m = k_1 + k_2 + \cdots + k_m$. Prove that, for each n, the interval $[s_n, s_{n+1})$ contains at least one perfect square.