2014 Science Atlantic Math Competition

Time: 3 hours.

Team members may collaborate with each other but not with others. Calculators and notes are forbidden.

Please write the answer to each question on a separate sheet (or sheets) of paper, and do not refer to other answers, as your answers to the various questions will be graded separately. Put your **team number** and the **question number** on ALL pages. Do not put your names, team name, or university on the answer sheets. Show all work.

Put your university, your own names, and your team number on the outside of the envelope before handing in your answers.

Few marks will be given for fragmentary or incomplete answers.

Each of the eight questions carries equal weight.

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QUESTIONS

1. A piece of paper is in the shape of a rectangle $ABCD$ with $AB = DC = 3$ and $AD = BC = 5$. The paper is folded so that A and C coincide. Find the length of the crease.

2. Find a polynomial P with integer coefficients such that $2^n + P(n)$ is divisible by 10 for all positive integers n, or show no such polynomial can exist.

3. Two points x_1 and x_2 are picked independently and at random on the interval [0,1]. The distribution is uniform: that is, the probability that each point is in a given interval $[a, b]$ is $b - a$. Find the probability that the points are within a distance $1/2$ of each other.

4. Find $\displaystyle\lim_{x\to 0} \frac{\tan(\sin^{-1}(x)) - \sin(\tan^{-1}(x))}{x^3}$.

5. Describe and sketch accurately $\{(x, y, z) : \ |x| + |y| \leq 1, \ |y| + |z| \leq 1, \ |z| + |x| \leq 1\}$.

6. Show that if a 5×5 matrix is filled with zeros and ones, there must always be a 2×2 submatrix (that is, the intersection of the union of two rows with the union of two columns) consisting entirely of zeros or entirely of ones.

7. A line segment of constant length 1 moves with one end on the x axis and the other end on the y axis. The region swept out - that is, the union of all possible placements - is R. Find the equation of the boundary of R.

8. Suppose that f is at least three times differentiable with $|f'''(x)| \leq 1$ on $[-1, 1]$, and $f(-1) = f(0) = f(1) = 0$. Find, with proof, the largest value that $|f(x)|$ can take on $[-1, 1]$.