2012 Science Atlantic Math Competition

QUESTIONS

1. Determine whether the series
$$\sum_{j=1}^{\infty}\left(\sum_{k=1}^{j} k\right)^{-1}$$
converges or diverges. If the series converges, find the value of the sum.

2. Solve the following system of equations:
$$\begin{aligned} xy - x - y &= 11 \\ yz - y - z &= 14 \\ zx - z - x &= 19 \end{aligned}$$

3. Let $U = \{(x, y) : x^2 + y^2 < 1\}$ be the open unit disc in the plane $\mathbb{R}^2$. A *chord* of U is naturally defined to be a chord of the unit circle with its distinct endpoints removed.

   Prove or disprove: there is a bijection $f : \mathbb{R}^2 \to U$ such that every straight line in $\mathbb{R}^2$ is mapped by f onto a chord of U.

4. Let $\lfloor x \rfloor$ denote the greatest integer less than or equal to x. Let $n = \left\lfloor \dfrac{1}{a - \lfloor a \rfloor} \right\rfloor$, for a positive non integer real number a. Show that $\lfloor (n + 1)a \rfloor \equiv 1(\text{mod n+1})$

5. Find all real values of a such that $x^3 - 6x^2 + 11x + a = 0$ has three integer solutions.

6. To see who pays for the beer, A and B play the following simple game. They shuffle a deck of cards, and then in turns draw cards. The first person to draw an ace pays for the beer. If A draws first, what is the probability that he buys? (Express your answer as a fraction in lowest terms.)

7. Suppose UV is a diameter of a circle, and that P and Q are points on the same semicircle with $UP < UQ$. The tangents to the semicircle at P and Q meet at R. Suppose that S is the point of intersection of UP and VQ. Prove that RS is perpendicular to UV.

8. Let $[L_n]$ be the $n \times n$ matrix with $[L_n]_{ij} = \dfrac{1 - (-1)^{\min\{i,j\}}}{2}$. Show that $[L_n]$ is invertible for all n and that every element of $[L_n]^{-1}$ is in $\{-1, 0, 1\}$.