2010 APICS Math Competition

Time: 3 hours.

Team members may collaborate with each other but not with others. Calculators and notes are forbidden.
Please write the answer to each question on a separate sheet (or sheets) of paper, and do not refer to other answers, as your answers to the various questions will be graded separately. Put your **team number** and the **question number** on ALL pages. Do not put your names, team name, or university on the answer sheets. Show all work.
Put your university, your own names, and your team number on the outside of the envelope before handing in your answers.
Few marks will be given for fragmentary or incomplete answers.
Each of the eight questions carries equal weight.

2010 APICS Math Competition

QUESTIONS

1. Given a (not necessarily isosceles) trapezoid $ABCD$ where AB is parallel to DC and $AD = DC = \frac{AB}{2}$, determine angle ACB.

2. In the country of Angina inflation is rampant and all prices are in powers of 10. The eccentricity of the President of Angina is such that currency is issued only in denominations having all digits equal to 2. i.e. \$2, \$22, \$222, \$2222, ...

   An Anginian buys a car for $\$10^{2010}$. What is the least number of notes that she could use?

3. Suppose that $a \neq 0$ and r are numbers such that ar^i is an integer for all $i \in \mathbb{N}$. Show that r is an integer. (Here, $\mathbb{N} = \{0, 1, 2, 3, \ldots\}$.)

4. For a square matrix A, call any matrix B such that $AB = A + B$ a friend of A. Call a nonzero vector $\mathbf{x}$ such that $A\mathbf{x} = \mathbf{x}$ a fixed point of A. Show that every square matrix has a fixed point or a friend but not both.

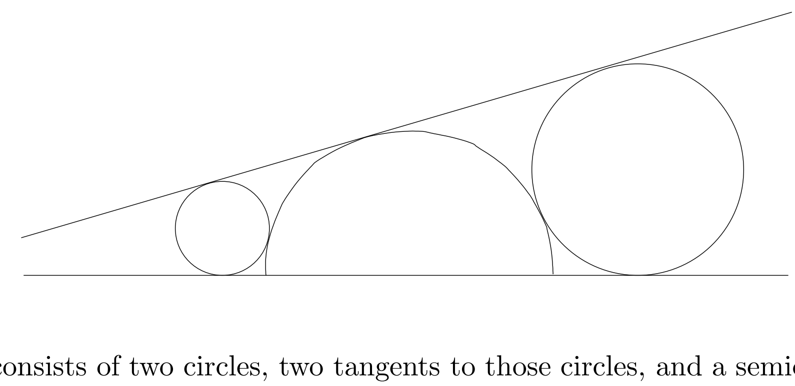
5. How many integer solutions are there to

$$a_0^2 + a_0a_1 + a_1^2 + a_1a_2 + ... + a_{2009}a_{2010} + a_{2010}^2 = 1?$$

6. Suppose a continuous function f has the property that $\underbrace{f(f(\ldots f(x)\ldots))}_{2010 f's} = 0$ for all x.

   Show that there must exist an interval $I = [a, b]$, $a \neq b$, with $0 \in I$ and $f(I) = 0$.

7. The pseudo-Smarandache function f is defined for positive integers n by: $f(n)$ is the smallest positive integer m such that $n|\frac{m(m+1)}{2}$. Prove that $f(n) \geq n$ if and only if $n = 2^r$ for some nonnegative integer r.

8. 

   The figure above consists of two circles, two tangents to those circles, and a semicircle whose diameter lies on one of the lines, and which is tangent to the other line and both circles. If the radii of the two circles are 4 and 9, find the radius of the semicircle.