2009 APICS Math Competition

Time: 3 hours.

Team members may collaborate with each other but not with others. Calculators and notes are forbidden.

Please write the answer to each question on a separate sheet (or sheets) of paper, and do not refer to other answers, as your answers to the various questions will be graded separately. Put your **team number** and the **question number** on ALL pages. Do not put your names, team name, or university on the answer sheets. Show all work.

Put your university, your own names, and your team number on the outside of the envelope before handing in your answers.

Few marks will be given for fragmentary or incomplete answers.

Each of the eight questions carries equal weight.

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QUESTIONS

1. Let x be an integer of the form $x = 111\ldots111$ with n "1s". Show that if x is prime, then n must also be prime.

2. Find a partition of the set {1,2,...,12} such that the largest integer in each subset is the sum of the other integers in that subset. Can the integers {1,2,..., 2009} be partitioned into subsets such that the largest integer in each subset is the sum of the other integers in that subset?

3. Find all continuous nondecreasing functions $f : \mathbb{R} \to \mathbb{R}$ such that $f \circ f = f$.

4. Prove that
$$\sin 20^\circ \sin 40^\circ \sin 60^\circ \sin 80^\circ = \frac{3}{16}.$$

5. Alice and Bob take turns filling in the entries of a 2009×2009 matrix with integers. Alice (who plays first) wins if the resulting matrix is singular, Bob (whose name is palindromic) wins if it is invertible. Which player has a winning strategy? What is that strategy?

6. Suppose that P and Q are antipodal vertices of a cubical box with edge length 1. (Thus PQ is a body diagonal for the cube.)

   (a) What is the radius of the largest sphere that can be put inside the box *without* being punctured by the diagonal PQ? (The diagonal should not meet the interior of the sphere.)

   (b) In how many different positions can such a sphere be placed in the box?

   (c) How many such spheres can simultaneously be put into the box?

7. Find all real solutions to the following system of equations:
$$\begin{aligned}(x+2y)(x+2z) &= 8\\ (y+2x)(y+2z) &= 9\\ (z+2y)(z+2x) &= 8\end{aligned}$$

8. For a real number x, $\lfloor x \rfloor$ is defined to be the greatest integer less than or equal to x. Show that
$$\lfloor n^{1/2} \rfloor + \lfloor n^{1/3} \rfloor + ... + \lfloor n^{1/n} \rfloor = \lfloor \log_2(n) \rfloor + \lfloor \log_3(n) \rfloor + ... + \lfloor \log_n(n) \rfloor$$
for $n \geq 2$.