2007 APICS Math Competition

Time: 3 hours.

Team members may collaborate with each other but not with others. Calculators and notes are forbidden.

Please write the answer to each question on a separate sheet (or sheets) of paper, and do not refer to other answers, as your answers to the various questions will be graded separately. Put your **team number** and the **question number** on ALL pages. Do not put your names, team name, or university on the answer sheets. Show all work.

Put your university, your own names, and your team number on the outside of the envelope before handing in your answers.

Few marks will be given for fragmentary or incomplete answers.

This question booklet has 8 questions over 2 pages. Each of the eight questions carries equal weight.

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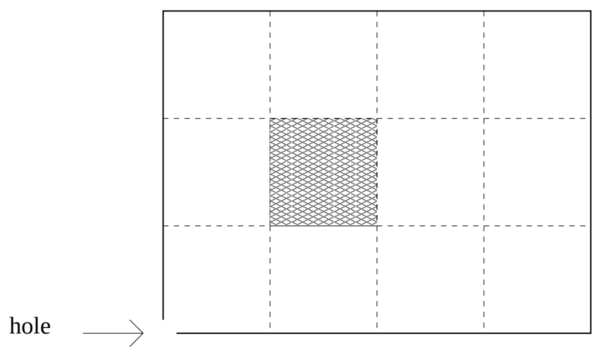
QUESTIONS

1. Scouts from a colony of ants discovered a wrapped chocolate bar. They ate a tiny hole in one corner of the foil wrapping, and then worker ants filed back and forth through the hole, carrying tiny morsels of chocolate back to their nest.

   The chocolate bar was a uniform rectangular slab, 3 cm by 4 cm, weighing 100 gm. The foil wrapping had 12 squares marked on it, as shown.

   Ants do not like to walk on foil so, as each ant entered the hole, it walked to the closest point on the chocolate surface - then removed a morsel. Ants hate walking on foil so much that efficiency drops as they keep walking over it. In fact, R, the rate at which the ants removed chocolate (gm/hour) changed with distance s (cm) from the hole to the chocolate surface according to the formula:

   $$R = 10e^{-s}.$$

   (a) As the last morsel of chocolate under the shaded square was removed, one of the ants announced: "We have now removed ____ gm of chocolate." Find the missing number.

   (b) How long did it take from the start of the project until removal of the last morsel of chocolate under the shaded square?

2. How many sequences $a_1, a_2, a_3, ..., a_n$ satisify these two conditions:

   (a) $a_i \in \{1, 2, 3\}$; and

   (b) $|a_{i+1} - a_i| = 1, i = 1, 2, \ldots, n-1$

3. In triangle ABC, D is located on BC so that AD is an altitude of the triangle. A point P is located on AD so that $\angle PCB = 30°$, $\angle PBA = 20°$, and $\angle PBC = 40°$. Find the sizes of $\angle PAC$ and $\angle PCA$.

4. Let a_0 and b_0 be positive real numbers. For $n = 0, 1, 2, \ldots$ let

$$a_{n+1} = \frac{(a_n + b_n/e)^2}{(a_n + b_n)^2} \quad ,$$

$$b_{n+1} = \frac{b_n \times (a_n + b_n/e)}{(a_n + b_n)^2} \quad .$$

Determine the values of

$$\lim_{n\to\infty} a_n \quad \text{and} \quad \lim_{n\to\infty} b_n \ .$$

5. The set $\mathcal{I}_3$ consists of all 3×3 invertible matrices with entries taken from the set $\{0, 1\}$. Is it possible to get from $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$ to $\begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix}$, by changing only one entry at a time, without leaving $\mathcal{I}_3$?

6. You have a large sheet of paper on your desk. There is a fixed point on your desk, which we will call the origin. Mark the point on the paper which is lying on the origin. Move the piece of paper 1cm away from you, and rotate it $\frac{180}{n}$ degrees clockwise around the origin. Move the piece of paper another 1cm away from you, and rotate it again by $\frac{180^\circ}{n}$. Repeat a total of n times. What is the position of the marked point on the paper relative to the origin?

7. Find all functions f defined on the integers such that

$$f(n^2) = f(n+m)f(n-m) + m^2 \ ,$$

for all integers m and n.

8. Factorize the polynomial $x^{256} + y^{256}$ completely over the real polynomials.