2005 APICS Math Competition

Time: 3 hours.

Team members may collaborate with each other but not with others. Calculators and notes are forbidden.

Please write the answer to each question on a separate sheet (or sheets) of paper, and do not refer to other answers, as your answers to the various questions will be graded separately. Put your fteam number and the **question number** on ALL pages. Do not put your names, team name, or university on the answer sheets. Show all work.

Put your university, your own names, and your team number on the outside of the envelope before handing in your answers.

Few marks will be given for fragmentary or incomplete answers.

This question booklet has 8 questions over 2 pages. Each of the eight questions carries equal weight.

QUESTIONS

1. An origami construction for an octagon works as follows.
   (a) Start with a square whose vertices are (in clockwise order) A, B, C, D.
   (b) Fold the square in half along the diagonal $\overline{AC}$ and unfold again.
   (c) Fold the edge $\overline{AB}$ onto the crease $\overline{AC}$ and leave it there temporarily.
   (d) Fold the vertex C to the current position of the vertex B and flatten. Label the endpoints of the crease thus made P_1, Q_1.
   (e) Unfold all creases.

   Do this three more times, creating creases $\overline{P_iQ_i}$ for $i = 2, 3, 4$.

   Prove that the eight points $\{P_1, Q_1, P_2, Q_2, P_3, Q_3, P_4, Q_4\}$, form the vertices of a regular octagon.

2. Show that there are no integers a, b, c such that $a^2 + b^2 - 8c = 6$.

3. For any triangle with angles α, β, γ prove that

$$\tan\frac{\beta}{2}\tan\frac{\gamma}{2} + \tan\frac{\gamma}{2}\tan\frac{\alpha}{2} + \tan\frac{\alpha}{2}\tan\frac{\gamma}{2} = 1$$

4. You are given eight identical $1 \times 1 \times 1$ cubes. Each cube has a pair of opposite (i.e. parallel) faces coloured red, a pair coloured green and a pair coloured blue. You put the cubes in a bag, shake them up and, blindfolded, assemble them into one $2 \times 2 \times 2$ cube. What is the probability that the big cube has each of its square faces coloured uniformly (eg. all red)?

5. A family of circles is defined inductively as follows. C_0 has radius 1. For $n > 0$, C_n is inscribed in a $n+2$-gon which in turn is is inscribed in C_{n-1}. Let r_n be the radius of C_n. Prove or disprove: $\lim_{n \to \infty} r_n = 0$

6. A circle has diameter $\overline{AB}$. P is a fixed point of $\overline{AB}$ lying between A and B. A point X, distinct from A and B, lies on the circumference of the circle.

   Prove that $\tan(\angle AXP)/\tan(\angle XAP)$ is constant for all values of X.

7. Show that the square root of a natural number of five or fewer digits never has a decimal part starting .1111, but that there is an eight-digit number with this property.

8. Let p be a permutation of the set $S_n = \{1, 2, 3, .., n\}$. An element j in S_n is called a fixed point of p if $p(j) = j$. Let $f(n)$ be the number of permutations having no fixed points and $g(n)$ be the number with exactly one fixed point. Show that $|f(n) - g(n)| = 1$.