APICS MATH CONTEST 2004

Attempt all questions. Few marks will be given for incomplete or fragmentary answers. Submit only one answer per question per team, working in collaboration.

Each question will be graded by a different person, so be sure to use a different sheet (or sheets) of paper for each question. Show all work; erasing or crossing out errors may improve your score.

To ensure objective grading, please do not identify your team on the problems sheet except by team number. Write your tem number and problem number on every sheet you submit. Write team number, university, and members' names on the envelope only.

No calculators, notes, or references are allowed. Team members may confer between each other but not with anybody else. Time: 3 hours.

(1) A fast food restaurant sells chicken nuggets in packs of 6, 9, and 20. This means that (for instance) you cannot buy exactly 11 nuggets. Is there a largest number N such that you cannot buy exactly N nuggets, and if so what is it? Prove your answer.

(2) Let A and B be four-digit base-ten palindromes and let C be a five-digit base-ten palindrome. If $A + B = C$, determine all possible values of C. (A *palindrome* is a number such as 13431 that reads the same forward or backward in positional notation with a specified base.)

(3) Find all pairs of positive integers (x, y) such that both $x^2 + 3y$ and $y^2 + 3x$ are perfect squares.

(4) Let $f : \mathbf{R} \to \mathbf{R}$ be continuous (where $\mathbf{R}$ represents the real numbers), with $f(f(f...f(x)) = x$ for all x, where the function f is iterated 2004 times.
(a) Find (with proof) $f(f(2004))$.
(b) Find (with proof) all possible values for $f(2004)$.

(5) Let x, y and z be complex numbers. Show that $x^n + y^n + z^n = (x + y + z)^n$ for all odd integers n, supposing this to be so when $n = -1$.

(6) The game of Square Meal is played with a heap of peanuts, initially containing N nuts. The two players take it in turns to eat a positive square number $(1, 4, 9, ...)$ of nuts. Whoever eats the last nut wins. For which values of N can the first player always win?

(7) Two spaces X and Y are said to be *isomorphic* if there are continuous maps $f : X \to Y$ and $g : Y \to X$ such that $fg(y) = y$ for all y and and $gf(x) = x$ for all x. Prove that $\mathbf{Q} \cap [0, 1)$ and $\mathbf{Q} \cap [0, \pi)$ are isomorphic, where $\mathbf{Q}$ is the rational numbers.

(8) Suppose that three circles in the plane are located so that each two intersects in two points, thereby giving a common cord to those two circles. Prove that these three chords pass through one point.