2003 APICS Mathematics Competition

October 17, 2003

INSTRUCTIONS:

- Present solutions to the following 8 questions in the booklets provided. Greater credit will be awarded to complete, well-written solutions.
- No notes, calculators, or other such aids are permitted.
- The competition is 3 hours in duration.

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1. Find all real solutions (x, y) to the following system.

$$\begin{aligned} x^2y + xy^2 &= 30 \\ \frac{1}{x} + \frac{1}{y} &= \frac{5}{6} \end{aligned}$$

2. The circle with equation $x^2 + y^2 = 1$ intersects the line $y = 7x + 5$ at two distinct points A and B. Let C be the point at which the positive x axis intersects the circle. Determine the angle ACB.

3. Added after the competition: Unfortunately, a disk of radius 1 cannot slide in the gap between adjacent coils of the spiral!

   An Archimedean spiral has the formula

$$r = \frac{1}{\pi}\,\theta\ .$$

   A disk of radius 1 can slide in the gap between adjacent coils of the spiral. Sketch the region the disk cannot reach and determine its area.

4. Write the positive integers in a spiral as follows.

$$\begin{array}{cccc} 7 & 8 & 9 & 10 \\ 6 & 1 & 2 & \cdots \\ 5 & 4 & 3 & \cdots \end{array}$$

   Place a white pawn on every square whose number is divisible by 3 or 5. Can a white queen, starting at 1, make a single move of a million squares? (A queen may move along any row, column, or diagonal for any distance, but may not move through a square occupied by a piece of her own colour.)

5. Let ABC be a right-angled triangle with right angle at C. Let CD be the altitude from C to AB. Let r_1 be the inradius of triangle ADC, let r_2 be the inradius of triangle BDC, and let r be the inradius of triangle ABC. Prove that $r_1 + r_2 \leq \sqrt{2}\, r$. When does equality occur?

   (The inradius of a triangle is the radius of the inscribed circle.)

6. Show that there is no function f such that

   (a) f is a one-to-one and onto mapping of R^2 to the open unit disk, and

   (b) the image of each line in R^2 is a chord of the unit disk.

7. At a Chinese restaurant, n people sit around a circular table and each person orders a different entree. The waiter brings out the entrees and puts one in front of each person. The table top can be rotated so that the entrees can be moved from one person to the next. (This is known as a "Lazy Susan".) Suppose that the entrees may be placed in such a way that no matter how the table is rotated, exactly one person will be matched up with his or her correct entree. Determine, with proof, all values of n for which this is possible.

8. Two mathematicians were surveying the damage done to Victoria Park in Charlottetown by Hurricane Juan. "It could have been worse," said one. "Less than one third of the trees were lost." His friend replied, "Yes, in fact if you multiply by 10 the number formed by taking the last two digits of the number of trees there used to be, and add to this the number formed by removing the last two digits of the number of trees there used to be, then you obtain the number of trees there is now." Not to be outdone, the first mathematician said "And if you take the number of trees that were lost, and reverse the order of the last two digits, and then insert a zero in front of the last two digits, then you get the number of trees that there used to be plus the number of trees that there are now."

   How many trees are now left in Victoria Park?