The Memorial University of Newfoundland
Department of Mathematics and Statistics

APICS 1999 Mathematics Competition

1. Find the volume of the solid formed by one complete revolution about the x–axis of the area in common to the circles with equations $x^2+y^2-4y+3 = 0$ and $x^2 + y^2 = 3$.

2. The Memorial University Philosophers' Jockey Club has just received the bronze busts of the ten members of their hall of fame. Each will be placed in its designated place on a single shelf, above the gold plaque bearing the name of the member. The ten busts are drawn at random from the crate. What is the probability that at no time will there be an empty space between two busts already placed on the shelf?

3. Prove that $\sin^2(x+\alpha) + \sin^2(x+\beta) - 2\cos(\alpha-\beta)\sin(x+\alpha)\sin(x+\beta)$ is a constant function of x.

4. In Scottish Dancing, there are three types of dances, two of which are fast rhythms, Jigs and Reels, and one is a slow rhythm, Strathspey.

   A Scottish Dance program always starts with a Jig. The following dances are selected (by type) according to the following rules:

   (i) the next dance is always of a different type from the previous one,

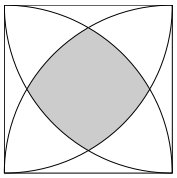
   (ii) no more than two fast dances can be consecutive.

   Find how many different arrangements of Jigs, Reels and Strathspeys are possible in a Scottish Dance list which has (a) seven dances, (b) fifteen dances.

5. Find all differentiable functions $f(x)$ which satisfy the integral equation

$$\big(f(x)\big)^{2000} = \int_1^x \big(f(t)\big)^{1999}\, dt\,.$$

6. Inside a square of side r, four quarter circles are drawn, with radius r and centres at the corners of the square.



Find the area of the shaded region.

7. Pat has a method for solving quadratic equations. For example, Pat solves $6x^2 + x - 2 = 0$ as follows:

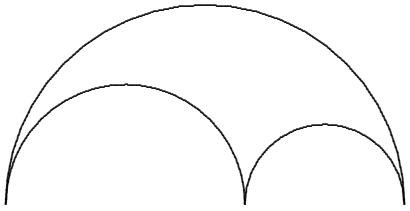
Step 1. Pat multiplies the leading coefficient by the constant coefficient and solved the simpler equation $x^2 + x - 12 = 0$ to get $(x + 4)(x - 3) = 0$.

Step 2. Pat then replaces each x by $6x$ (x times the leading coefficient) to get $(6x + 4)(6x - 3) = 0$.

Step 3. Pat then simplifies this equation to get $(3x + 2)(2x - 1) = 0$, which solves the original equation.

Prove or disprove that Pat's method always works.

8. An arbelos consists of three semicircular arcs as shown:



A circle is placed inside the arbelos so that it is tangent to all three semicircles.

Suppose that the radii of the two smaller semicircles are a and b, that the radius of the circle is r.

Assuming that $a > b > r$ and that a, b and r are in arithmetic progression, calculate a/b.