1998 APICS MATHEMATICS CONTEST

Saint Mary's University
October 16, 1998, 3 p.m. - 6 p.m.

Rules:

- Teams of two are to work in cooperation and to submit *one* set of answers each.
- No notes, calculators, or other such aids are permitted.
- You may not communicate with noncontestants (except invigilators) or other teams.
- There are 9 questions.

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1. Fred and Cathy play the following game. They are given the polynomial $f(x) = ax^3 + bx^2 + cx + d$. They take turns, Cathy first, in replacing a, then b, then c and finally d with positive integers. Fred wins if the resulting polynomial has at least two distinct roots. Who should win and what is the winning strategy?

2. Define the integer sequence $\{T_n\}$ by $T_0 = 0$, $T_1 = 1$, $T_2 = 2$ and $T_{n+1} = T_n + T_{n-1} + T_{n-2}$ ($n \geq 2$).

   Compute

   $$S := \sum_{n=0}^{\infty} \frac{T_n}{2^n}.$$

3. Let $X_1, X_2, \ldots, X_n$ be independent, integer valued random variables with p = Probability$\{X_k$ is even$\}$. Form the sum S_n of the random variables. Show that the probability that the sum is even is

   $$[1 + (2p-1)^n] / 2.$$

4. Show that there do not exist four points in the Euclidean plane such that the pairwise distances between them are all odd integers.

5. If $\{a_n\}$ is a sequence of positive integers such that

   $$\lim_{n\to\infty} \frac{a_n}{a_1 + a_2 + \ldots + a_n} = 0,$$

   show that there is a sequence $\{b_n\}$ of positive integers such that for every positive integer $n \geq 2$

   $$\frac{b_n}{b_1 + b_2 + \ldots + b_n} \leq \frac{1}{3}$$

   and for some positive integer N we have $a_n = b_n$ for all $n \geq N$.

6. For $a > 1$ evaluate

$$\int_0^a x a^{(-\lfloor \log_a x \rfloor)} dx,$$

where $\lfloor t \rfloor$ denotes the greatest integer less than or equal to *t*.

7. Let *ABCD* be a cyclic quadrilateral, inscribed in a circle ω. Let *A*', *B*', *C*', *D*' be the points where the tangents at *A* and *B*, at *B* and *C*, at *C* and *D* and at *D* and *A*, respectively, intersect. Prove that the lines *AC*, *BD*, *A*'*C*' and *B*'*D*' are concurrent, that is, they intersect at one point.

8. The expression

$$\underbrace{(\ldots(((x-2)^2-2)^2-2)^2-\ldots-2)^2}_{n\,-\,\text{times}} \qquad (1)$$

is multiplied out and coefficients of equal powers are collected. Find the coefficient of x^2.

9. Let $f(n) = 2n^2 + 14n + 25$. We see that $f(0) = 25 = 5^2$. Find two positive integers *n* such that $f(n)$ is a perfect square.