APICS Mathematics Contest 1996

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1. Simplify $\dfrac{\sqrt{\sqrt{5}+2}+\sqrt{\sqrt{5}-2}}{\sqrt{\sqrt{5}+1}}$ into the form $\dfrac{a+b\sqrt{c}}{d}$ where *a,b,c,* and *d* are integers.
2. Find all real solutions to the simultaneous equations
$$\begin{aligned} x^2y+xy^2 &= 1 \\ x^3+y^3 &= 5\,. \end{aligned}$$
3. The lengths of the sides of a triangle are 3,4, and 5 units. Prove that there is exactly one straight line which simultaneously bisects the area and the perimeter of the triangle.
4. Let $X_1,\ldots,X_n$ be independent uniform $[0,1]$ random variables. Let
$$S_n = X_1 + \ldots + X_n.$$
Determine the distribution of $S_n - [S_n]$, where $[x]$ is the greatest integer less than or equal to *x*.
5. A certain number of 0's, 1's, and 2's are written on a blackboard. Two unequal digits are erased and the third digit is written in their place (e.g., write 2 if you erase 0 and 1.) This operation is repeated until no two unequal digits remain on the blackboard. Show that if only a single digit remains, that digit is independent of the order in which the digits were erased.
6. Let *F* be a nondecreasing real function (if $x \geq y$ then $F(x) \geq F(y)$) defined on $[0,1]$, such that
$$\begin{aligned} F\left(\frac{x}{3}\right) &= \frac{F(x)}{2} \\ F(1-x) &= 1-F(x)\,. \end{aligned}$$
Find $F(300/1996)$ and $F(1/13)$.
7. Show that if $f(x) = \int_0^x \cos\frac{1}{t}\,dt$ for $x \neq 0$ and $f(0) = 0$, then $f'(0) = 0$.
8. Suppose that *f* and *g* are functions $R \to R$ such that $f(g(x)) = x$ and $f(f(f(f(x)))) = 8f(x)$ for all *x*. What values can $g(1996)$ have? (Note R represents the real numbers.)