# APICS Mathematics Contest 1995

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1. Given the functions $g : R \to R$ and $h : R \to R$, with $g(g(x)) = x$ for every $x \in R$, and $a$ a real number such that $|a| \neq 1$, prove taht there exists exactly one function $f : R \to R$ such that

$$a\, f(x) + f(g(x)) = h(x)$$

for every $x \in R$.

2. A solid fence encloses a square field with sides of length $L$. A cow is in a meadow surrounding the field for a large distance on all sides, and is tied to a rope of length $R$ attached to a corner of the fence. What area of the meadow is available for the cow to use?

3. Find all solutions to

$$(x^2 + y)(x + y^2) = (x - y)^3$$

where $x$ and $y$ are integers different from zero.

4. For what positive integers $n$ is the n$th$ Catalan number,

$$\frac{1}{n+1}\begin{pmatrix} 2n \\ n \end{pmatrix},$$

odd?

5. $N$ pairs of diametrically opposite points are chosen on a circle of radius $1$. Every line segment joining two of the $2N$ points, whether in the same pair or not, is called a diagonal. Show that the sum of the squares of the lengths of the diagonals depends only on $N$; and find that value.

6. A finite pattern of checkers is placed on an infinite checkerboard, at most one checker to a square; this is Generation $0$. Generation $N$ is generated from Generation $N$-$1$ (for $N = 1, 2, 3, \ldots$) by the following process: if a cell has an odd number of immediate horizontal or vertical neighbours in Generation $N$-$1$, it contains a checker in Generation $N$; otherwise it is vacant.

   Show that there exists an $X$ such that Generation $X$ consists of at least **1995** copies of the original pattern, each separated from the rest of the pattern by an empty region at least 1995 cells wide.

7. $A$ and $B$ play a game. First $A$ chooses a sequence of three tosses of a coin and tells it to $B$; then $B$ chooses a different sequence of three tosses and tells it to $A$. Then they throw a fair coin repeatedly until one sequence or the other shows up as three consecutive tosses.

   For instance, $A$ might choose (head, tail, head); then $B$ might choose (tail, head, tail). If the sequence of tosses is (head, tail, tail, head, tail), $B$ would win.

   If both players play rationally (make their best possible choice), what is the probability that $A$

wins?