APICS Mathematics Contest 1994

1. If $a + b + c + d = 0$ where a,b,c,d are integers, prove that

$$S = a^5 + b^5 + c^5 + d^5$$

is divisible by **10**. [All the exponents are 5]

2. The rectangle $ABCD$ is given. There are points K,L,M,N distinct from the vertices, on the sides AB,BC,CD and DA, respectively, such that KL is parallel to MN and KM is perpendicular to NL. Prove that the point of intersection of KM and LN lies on the diagonal BD of the rectangle.

3. Let a_0 be any natural number expressed in its decimal representation. Form a sequence of numbers $a_1, a_2, \ldots, a_n, \ldots$ according to the following rule:

If the last digit of a_n does not exceed 5, then delete this digit and let the remaining sequence of digits be the decimal representation of a_{n+1} (if a_{n+1} does not contain any digits, then the process terminates).

Otherwise, let $a_{n+1} = 9a_n$. Can an initial value a_0 be found for which the resulting sequence is infinite?

4. Prove that $\tan^2\left(\frac{\pi}{7}\right) + \tan^2\left(\frac{2\pi}{7}\right) + \tan^2\left(\frac{3\pi}{7}\right) = 21.$

5. Let $f(x)$ be any continuous solution to the functional equation

$$f(f(x)) + f(x) = 2x + 12.$$

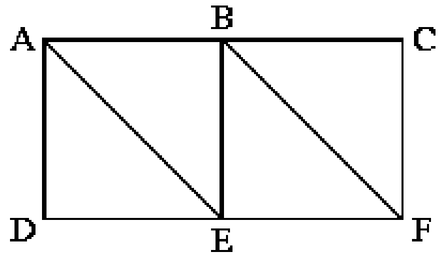
Show that $x + 3 < f(x) < x + 6$.

6. There is a fortune-teller at the annual village fair. Nine fortune-tellers out of ten are always right; the other ones are always wrong. Nobody at the fair, of course, ever knows which sort is there this year! Another annual feature of the fair is the lottery. Each of the twenty boys in the village has an equal chance to be chosen as the Harvest King; and each of the twenty girls has an equal chance to be chosen as the Harvest Queen.

   (i) Henry went into the fortune-teller's tent. He asked the fortune-teller whether he would be chosen, and was told that he would. What are his chances now?

   (ii) As he came out, Henry met his friend Anne. He told her about the prediction. She immediately went into the tent and asked whether she would be chosen as Harvest Queen. The fortune-teller told her that she would be chosen; what are her chances now?

7. The map of a village of dwarves is as follows



and only dwarves whose houses are directly joined by a path consider themselves neighbours. For instance, ***A*** and ***B*** are neighbours, but ***A*** and ***C*** are not.

One day, one of the dwarves is bored. He composes a chain letter, which says ``If, on any day, you get an odd number of copies of this letter, you must send a copy to all your neighbours. If you do not do this, your beard will fall off." He sends one copy to all of his neighbours. Prove that the chain letter will never stop circulating.

8. To play the game FADEOUT, you begin by writing any set of **1994** numbers on the blackboard. You are then permitted to erase any pair of them, say ***a*** and ***b***, and replace them by the number $(a+b)/4$. You continue this process **1993** times, after which only one number remains on the blackboard.

   Suppose that at the beginning of the game, all **1994** numbers on the board are **1**. Prove that the number left at the end will never be less than $1/1994$.