APICS Mathematics Contest 1992

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1. Show that for positive integers n,

$$n^4 + 3n^2 + 1$$

is never a perfect square.

2. Find all triples of real numbers (x, y, z) satisfying the equations

$$\frac{3xy}{x+y} = 5 \qquad \frac{2xz}{x+z} = 3 \qquad \frac{yz}{y+z} = 4.$$

3. If two numbers a,b are selected at random independently from the interval $[0, 1]$ so that each number has the same chance of being selected (i.e., the interval has the uniform probability distribution), what is the probability that $x^2 + ax + b$ has real roots?

4. Starting with two positive integers S_0, S_1, define a sequence as follows. For $n \geq 2$, if S_{n-1} is even, then $S_n = S_{n-1}/2$; otherwise, $S_n = S_{n-1} + S_{n-2}$. Thus, if $S_0 = 2,\ \ S_1 = 5$, the sequence begins $(2, 5, 7, 12, 6, 3, 9, \ldots)$. Show that for $S_0, S_1 < 1992$, the sequence always begins to cycle at or before the thirteenth terms; that is, there exists $p > 0$ such that for $n \geq 13,\ \ S_n = S_{n+p}$.

5. Find non-consecutive integer solutions to

$$1992^{50} + x^2 + 1 = 1992^{26} + 1992^{24} + y^2.$$

6. Let $A = (a_{ij})$ be an $m \times n$ matrix of mn distinct numbers for some $m, n \geq 2$. Rearrange each row of A so that the numbers are in increasing order from left to right. Call the resulting matrix A'. Now rearrange each column of A' so that the numbers are in increasing order from top to bottom. Let A'' be the resulting matrix. Show that each row of A'' is in increasing order from left to right.

7. How many symmetric $n \times n$ matrices can be constructed with the following properties?

   (a) All of the elements are either 0 or 1.
   (b) The elments on the main diagonal are 0.
   (c) Exactly two of the elements in each row are 1's.

8. For points $P = (a, b),\ \ Q = (c, d)$ in the plane, write $P \leq Q$ if $a \leq c$ and $b \leq d$. Fix two points A and B in the plane. Let S be any subset of the plane which contains both A and B. Show that for any $P \in S$ there is a $Q \in S$ satisfying $Q \leq P$ and also satisfying at least one of the following conditions:

(i) $Q \leq A$ and $Q \leq B$

(ii) There is no $R \in S$ such that $R \leq A$ and $R \leq Q$

(iii) There is no $R \in S$ such that $R \leq B$ and $R \leq Q$