APICS Mathematics Contest 1986

1. Find the largest and the smallest values of $3\sin^2 x + 2\sin 2x$.

2. How many zeroes are used in writing all of the numbers from 0 to $2^n - 1$ (inclusive) in binary form?

3. Let $p_1(x,y)$ and $p_2(x,y)$ be two polynomials in the variables x and y. A simultaneous zero of the polynomials is defined to be an ordered pair (x_*, y_*) of real or complex numbers such that $p_1(x_*, y_*) = 0$ and $p_2(x_*, y_*) = 0$. Determine the number of simultaneous (not necessarily distinct) zeroes for the polynomials:

$$\begin{aligned} p_1(x,y) &= x^2 + 3xy + 2y^2 + x - y + 2 \\ p_2(x,y) &= x^2 - xy^2 + 2xy - 7x + 4y^4 + y - 1 \ . \end{aligned}$$

4. Determine a_1, a_2, a_3 so that the function:

$$f(x) = a_1 \cdot |x - b_1| + a_2 \cdot |x - b_2| + a_3 \cdot |x - b_3|$$

where $b_1 < b_2 < b_3$ are given, satisfies the conditions: $f(b_2) = c \neq 0$, and $f(x) = 0$ for $x \le b_1$ and $x \ge b_3$.

5. For integers $n > 2$ and real numbers $s > 0$, show that

$$\left(\prod_{i=0}^{n}(s+i)\right)\left(\sum_{j=0}^{n}\frac{1}{s+j}\right) < (n+1)\prod_{k=1}^{n}\left(s+k-\frac{1}{2}\right).$$

Note that $\prod_{k=0}^{n} a_k = a_0 a_1 \ldots a_n$.

6. Find with justification the smallest value of n for which $a_n = 6^n + 8^n$ will be a multiple of 49.

7. Given a function $g(x)$, define $m_1(x) = g(x)$ and

$$m_{n+1}(x) = \min_{0 \le t \le x}[m_n(t) + g(x-t)]$$

where $n = 1, 2, \ldots$. Find $\lim_{n\to\infty} m_n(x)$ if $g(x) = x^2$.