APICS Mathematics Contest 1983

1. If n is a positive integer, show that $\sum_{k=0}^{n} \binom{n}{k}^2 = \binom{2n}{n}$.

2. Given numbers x,y,z such that

   $$x + y + z = 1,\ \ x^2 + y^2 + z^2 = 2,\ \ x^3 + y^3 + z^3 = 3.$$

   Compute $x^4 + y^4 + z^4$.

3. An ant starts at a point P on the bottom edge of a right circular cylinder of radius R and height H. If the ant makes n complete circuits around the cylinder and finishes at a point at the top edge directly above its starting point, find, with justification, the length of its shortest possible path.

4. Let f be an integrable function and let $F_1(x) = \int_0^x f(t)dt,\ \ F_n(x) = \int_0^x F_{n-1}(t)dt,\ \ n \geq 2$. Show that $F_n(x) = \int_0^x \frac{(x-t)^{n-1}}{(n-1)!} f(t)dt$ for $n \geq 1$.

5. Given a surface S defined by $f(x,y,z) = 0$ such that (a) the intersection of S with any plane z = constant is the curve xy = constant and (b) the intersection of S with any plane x = constant is the curve $\frac{y}{z}$ = constant. Find the equation of the surface S.

6. Determine the locus (path) of the point O of intersection of the altitudes (orthocentre) of a triangle ABC, if the locus of vertex A is a line parallel to BC.

7. Show that $\sum_{n=0}^{\infty} \frac{1}{(n!)^2}$ is irrational.

8. Select a non-negative integer n at random. What is the probability that the first digit of 2^n is a ``one'' (in base 10 notation)?