APICS Mathematics Contest 1982

1. Show that $(\sin x + \cos x)^4 \leq 4$.

2. If *a* and *b* are positive integers, find the probability that $(a^2 + b^2)/5$ is a positive integer.

3. For a positive value of *c*, the limit

$$L = \lim_{x \to +\infty} x^c e^{-2x} \int_0^x e^{2t} \sqrt{et^2 + 1}\, dt$$

exists, is finite an nonzero. Find this value of *c*, and the limit.

4. How many real roots does the function

$$f(x) = 1 + x + \frac{x^2}{2} + \ldots + \frac{x^n}{n}$$

have?

5. Show that every integer *x* can be expressed uniquely in the form

$$x = \sum_{k=1}^{m} a_k k!$$

where $0 \leq a_k \leq k$.

6. Given two disjoint finite sets *A* and *B* in the plane. Suppose that the line segment joining any two points of *A* contains a point of *B* and the line segment joining any two points of *B* contains a point *A*. Show that all the points of $A \cup B$ lie on a straight line.

7. Given a triangle $\triangle ABC$ and a straight line ℓ, find the point *P* on ℓ such that $(PA)^2 + (PB)^2 + (PC)^2$ is the smallest.

8. For $k \geq 0$, let *S* be the set of all numbers of the form

$$s = \sqrt{k \pm \sqrt{k \pm \ldots \pm \sqrt{k}}}$$

with arbitrary finite sequence of signs. Show that if $k \geq 2$, then all $s \subset S$ are real and if *k* = 2, *S* is dense in (0.2).