APICS Mathematics Contest 1978

1. The expression of a positive integer, n in base b is

$$n = 1254_b.$$

It is known that the expression of the integer $2n$ in the same base is

$$2n = 2541_b.$$

Determine the values of b and n in base **10**.

2. Let A be an $n \times n$ matrix of **0**'s and **1**'s such that there are exactly r **1**'s in each row $(0 < r < n)$ and any two rows have common **1**'s in exactly s columns $(0 < s < r)$. By considering $A^t A$, or otherwise, show that A is invertible.

3. Find all integers $n \geq 1$ such that $\binom{n}{k}$ is odd for all k, $0 \leq k \leq n$.

4. Find all the solutions of the equation

$$\sin k\theta = \sin\theta, \quad 0 \leq \theta \leq 2\pi.$$

5. Evaluate the infinite product: [the exponents are 1/3,1/9,1/27,1/81]

$$\left(\frac{1}{2}\right)^{\frac{1}{3}} \left(\frac{1}{4}\right)^{\frac{1}{9}} \left(\frac{1}{8}\right)^{\frac{1}{27}} \left(\frac{1}{16}\right)^{\frac{1}{81}} \cdots$$

6. Suppose that a finite set of regular hexagonal tiles is placed on the plane. Show that the tiles can be coloured with **4** colors in such a way that no **2** tiles sharing the same edge are the same colour.

7. For a real number x, let $\{x\}$ denote $x - [x]$, where $[x]$ is the largest integer less than x. Show that:

$$\lim_{n \to \infty} \{(2 + \sqrt{3})^n\} = 1.$$