2025 Science Atlantic Math Competition

Time: 3 hours.

Contestants will work in teams of two. Team members may collaborate but no other collaboration is permitted. Calculators, mathematical software, books, notes, etc are forbidden.

Please write the answer to each question on a separate sheet (or sheets) of paper, and do not refer to other answers, as your answers to the various questions will be graded separately.

Do put your *team number* and the *question number* on every page submitted. Do *not* put your name or university on the answer sheets. Show all work.

Each of the eight questions carries equal weight.

QUESTIONS

There are eight questions, over two pages. Not all questions are of equal difficulty.

1. For which positive integers n, if any, does the series

$$\sum_{j=0}^{\infty} \lfloor \frac{n+2^j}{2^{j+1}} \rfloor$$

converge? Determine the value when the series does converge.
Note: $\lfloor x \rfloor$ is the greatest integer less than or equal the real number x.

2. Suppose A is a point exterior to the unit sphere $\mathcal{S}$ in $\mathbb{R}^3$. The tangents from A to $\mathcal{S}$ form a right circular cone $\mathcal{K}$. (For our purposes this is a finite cone ending at the points of tangency. Note that the usual base of the cone is not included here.)

   At what distance should A be from the centre of the sphere so that $\mathcal{S}$ and $\mathcal{K}$ have equal surface areas?

3. Let

$$f(x) = x^4 + x^3 + x^2 + x^1 + 1.$$

Find the remainder when $f(x^5)$ is divided by $f(x)$.

More questions over ...

4. Sam goes into the gym and puts a marble of radius 1cm in the corner, touching two walls and the floor. (Assume all surfaces are flat and the angles are right.) A second student comes along and puts a ball in the corner so that it just touches both walls, the floor, and the marble. Several more students do the same, each ball touching the walls, the floor, and the previous ball.

   What is the the first ball radius that is greater than or equal to 1m? Give your answer in the form
$$\frac{a + b\sqrt{c}}{d},$$
   where a, b, c, d are integers.

5. Late in the year 1 CE, the villagers of Marzipan invented a winter solstice tradition: each household sent a fruitcake to one other household. None of these fruitcakes were ever eaten: instead, each year afterward, each household passed their fruitcake on to the same household that they had given one to in the previous year. This tradition has lasted over the centuries, and this year (2025 CE), for the first time, every household will get its original fruitcake back.

   What is the smallest possible number of households in the village of Marzipan?

6. Let $S(n)$ be the set of ordered quadruples (a, b, c, d) from $\{1, \ldots, n\}$ such that $a, b, c < d$. Does there exist n such that $S(n)$ has exactly 2025 elements?

7. For each natural number n,define an $n \times n$ matrix $P(n)$ with $(P(n))_{ij} = \binom{n+i-2}{j-1}$ for $i, j = 1, 2, \ldots, n$. Find (with proof) $\det(P(n))$.

8. You are given positive integers p, q, r, s satisfying
$$qr - ps = 1.$$
   Suppose x, y are positive integers with
$$\frac{p}{q} < \frac{x}{y} < \frac{r}{s}.$$
   Find the smallest value of y satisfying this last condition and determine all x that go with it.