Science Atlantic Mathematics Contest
October, 2024

- Each team (of at most two students) should submit solutions to as many problems as possible. You may share or divide up the investigation and writing up of problems as you see fit.
- Communication with other teams, or use of notes, phones, books, computers, etc. is forbidden.
- Few points (if any) will be given for incomplete solutions. Calculations and proofs should be clearly and fully set out.
- Put your team ID number onto each answer sheet. Do not put your personal names, team name, university, or any other identification except that number onto your submission. The top of a typical page submitted should be similar to

  Question 3; page 1 of 2. Team 97

- Put each solution on a separate sheet (or sheets as needed). Different questions will be graded by different people. Do not put answers to different quesions on the same sheet.
- The contest lasts 3 hours.

1. Clearly describe all natural numbers m such that 13 divides $3^m + m$.

2. Suppose C_1, C_2 are non-intersecting circles in the plane, neither inside the other. The tangents to C_2 from the centre of C_1 meet C_1 in points A, B. Likewise, the tangents to C_1 from the centre of C_2 meet C_2 in points C, D. Prove that the segments AB and CD have the same length.

3. Let S be a set, and $\diamond$ a binary operation on S such that

   - $x \diamond x = x$ for all $x \in S$
   - $(x \diamond y) \diamond z = (y \diamond z) \diamond x$ for all $x, y, z \in S$.

   Show that $x \diamond y = y \diamond x$ for all $x, y \in S$.

4. Determine, with proof, the value of

$$\lfloor \frac{\lfloor nx \rfloor}{n} \rfloor$$

   for each positive integer n and real number x.
   (Recall that $\lfloor t \rfloor$ denotes the largest integer which does not exceed t.)

5. Let A_n be the symmetric $n \times n$ matrix with entries $a_{i,i} = 2$, for $1 \leq i \leq n$; $a_{i,i+1} = a_{i+1,i} = -1$, for $1 \leq i \leq n-1$; $a_{i,j} = 0$, otherwise. Describe, with proof, all those n for which the determinant of A_n is a prime number.

6. Let $T(n)$ be the number of subsets of $\{1, 2, ..., n\}$ which do not contain three consecutive integers. Show that there are positive constants k and α such that for all large enough integers n, $T(n)$ is the integer closest to the real number $k\alpha^n$. (This is often written as $\{k\alpha^n\}$.)

7. If $f : \mathbb{R} \to \mathbb{R}$ is an infinitely differentiable real function obeying

$$f'(x) = f(x^2), f(1) = 1,$$

   for how many $n \in \{1, 2, ..., 2024\}$ is $f^{(n)}(0) = 0$?

8. For positive integers a, n, let $a \uparrow 1 = a$, and recursively define

$$a \uparrow (n+1) = a^{(a \uparrow n)}.$$

   What is the least integer n such that $3 \uparrow n > 9 \uparrow 2024$?