2022 Science Atlantic Math Competition

Time: 3 hours.

Contestants will work in teams of two. Team members may collaborate but no other collaboration is permitted. Calculators, mathematical software, books, notes, etc are forbidden.

Please write the answer to each question on a separate sheet (or sheets) of paper, and do not refer to other answers, as your answers to the various questions will be graded separately.

Do put your *team number* and the *question number* on every page submitted.

Do *not* put your name or university on the answer sheets. Show all work.

Each of the eight questions carries equal weight.

QUESTIONS

There are eight questions, over two pages. Not all questions are of equal difficulty.

1. Determine, with proof, all real numbers m such that there exists a continuous function $f : \mathbb{R} \to \mathbb{R}$ satisfying $f(f(x)) = 1 + mx$ for all $x \in \mathbb{R}$.

2. Find, with proof, all real solutions to
$$x2^{1/x} + \frac{1}{x}2^x = 4.$$

3. For what integer values of a is $n(n+5)(2n+a)$ divisible by 6 for every natural number n?

4. Let ω be a plane in 3-space which passes through a vertex A of a unit cube. For the vertices B_1, B_2, B_3 of the cube which are adjacent to A, let F_1, F_2, F_3 in turn be the feet of the perpendiculars dropped to ω. Determine the value of
$$(AF_1)^2 + (AF_2)^2 + (AF_3)^2.$$

5. Recall that the *population standard deviation* is defined to be

$$\sigma := \sqrt{\frac{\sum_{i=1}^{n}(\mu - x_i)^2}{n}}$$

where μ is the mean of the x_i.
The polynomial

$$a_0 + a_1 x + \cdots + a_{2022}x^{2022}$$

has 2022 roots $\{x_i : 1 \leq i \leq 2022\}$ (not necessarily distinct.) Find μ and σ for these 2022 values.

6. Find all triples (x, y, z) of non-negative real numbers satisfying the system of equations

$$\begin{aligned} \sqrt[3]{x} - \sqrt[3]{y} - \sqrt[3]{z} &= 16 \\ \sqrt[4]{x} - \sqrt[4]{y} - \sqrt[4]{z} &= 8 \\ \sqrt[6]{x} - \sqrt[6]{y} - \sqrt[6]{z} &= 4. \end{aligned}$$

7. Give a closed-form expression for the function

$$f(x) = \lim_{n\to\infty}\left[\lim_{m\to\infty}\cos^m(n!\pi x)\right],$$

where m, n are natural numbers.

8. Suppose $p(x, y)$ is a polynomial in two variables and with real coefficients such that

(a) $p(x, y) = p(y, x)$;

(b) $x - y$ is a factor of $p(x, y)$.

Prove that $(x - y)^2$ is a factor of $p(x, y)$.