Science Atlantic Math Problem Solving Contest, October, 2021

Each team (of at most two students) should submit joint solution to as many problems as possible.
Communication with other people, or use of notes, phones, books, computers, etc is forbidden.
Few points (if any) will be given for incomplete solutions.
Put your team ID number onto each sheet. Do not put your personal names, team name, university, or any other identification except that number onto your submission.
Put each solution onto a separate sheet or sheets. Different questions will be graded by different people, most likely in different cities!
At the end of the allotted time **(three hours)** you must immediately give your papers to the local invigilator. You will not email your own solutions.

1. Let us call an integer *good* if it can be written as a difference of two square numbers. (A square number is a number of the form n^2 for some integer n.) For instance, 7 is good since $7 = 4^2 - 3^2$, but,as you might discover, 6 is not good.
   Which integers are good?

2. $\triangle ABC$ is an arbitrary triangle with area 1. The edge AB is extended past B to a point B' such that $|BB'| = |AB|$. Similarly, the edge BC is extended past C to a point C' such that $|CC'| = 2|BC|$; and CA is extended past A to a point A' such that $|AA'| = 3|CA|$. Find the area of $\triangle A'B'C'$.

3. Define a "Fibonacci-like" sequence as follows: $A_1 = A_2 = 1$, and $A_n = 2A_{n-2} + A_{n-1}$; so $A_3 = 2 \times 1 + 1 = 3, A_4 = 2 \times 1 + 3 = 5$, and so on. Prove that for odd n,
$$\sum_{i=1}^{n-1} A_i = A_n - 1$$

4. Prove that there is no integer solution to
$$p^2 = \frac{q(q+1)}{2} = \frac{3r^2 - r + 4}{2}.$$

5. Given that $\sin(xy) = 1$, find the least upper bound of $\sin(x)\sin(y)$, and show that this is never achieved.

6. Find, with proof, $\int_0^{\pi/2} \cos^{31416}(x)dx$.

7. Ruby and Sapphire are celebrating Pi Day by sharing a circular pie. Ruby has two red birthday cake candles, and Sapphire has two blue candles. Ruby starting, they will alternately place one candle on the perimeter of the pie. (Of curse, no two candles may be in the same place!) After all the candles are placed, each girl will get the portion of the pie that is closer to one of her candles than to any of the others. The goal is to get strictly more pie than one's opponent; an equal division is a draw.

   Either find a winning strategy for one player and show that it is essentially unique, or show that the game, rationally played, is a draw.

8. Fifteen sheets of paper of various sizes and shapes lie on a desktop, covering it completely. The sheets may overlap one another and may even hang over the edge of the desktop. Prove that five of the sheets can be removed so that the remaining ten sheets cover at least two-thirds of the desktop.